ON WEAK FANO VARIETIES WITH LOG CANONICAL SINGULARITIES

YOSHINORI GONGYO

ABSTRACT. We prove that the anti-canonical divisors of weak Fano 3-folds with log canonical singularities are semiample. Moreover, we consider semiampleness of the anti-log canonical divisor of any weak log Fano pair with log canonical singularities. We show semiampleness dose not hold in general by constructing several examples. Based on those examples, we propose sufficient conditions which seem to be the best possible and we prove semiampleness under such conditions. In particular we derive semiampleness of the anti-canonical divisors of log canonical weak Fano 4-folds whose lc centers are at most 1-dimensional. We also investigate the Kleiman-Mori cones of weak log Fano pairs with log canonical singularities.

1. INTRODUCTION

Throughout this paper, we work over $\mathbb{C}$, the complex number field. We start by some basic definitions.

Definition 1.1. Let X be a normal projective variety and Δ an effective $\mathbb{Q}$-Weil divisor on X. We say that (X, Δ) is a *weak log Fano pair* if $-(K_X + \Delta)$ is nef and big. If $\Delta = 0$, then we simply say that X is a *weak Fano variety.*

Definition 1.2. Let X be a normal variety and Δ an effective $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor. Let $\varphi : Y \to X$ be a log resolution of (X, Δ). We set

$$K_Y = \varphi^*(K_X + \Delta) + \sum a_i E_i,$$

where E_i is a prime divisor. The pair (X, Δ) is called

(a) *kawamata log terminal* (*klt*, for short) if $a_i > -1$ for all i, or
(b) *log canonical* (*lc*, for short) if $a_i \geq -1$ for all i.

Definition 1.3 (Lc center)**.** Let (X, Δ) be an lc pair. We call that $C \subset X$ is an *lc center* of (X, Δ) if there exists a log resolution $\varphi : Y \to X$ such that $\varphi(E) = C$ for some prime divisor E on Y with $a(E, X, \Delta) = -1$.

There are questions whether the following fundamental properties hold or not for a log canonical weak log Fano pair (X, Δ) (cf. [S, 2.6. Remark-Corollary], [P, 11.1]):

(i) Semiampleness of $-(K_X + \Delta)$.
(ii) Existence of $\mathbb{Q}$-complements, i.e., existence of an effective $\mathbb{Q}$-divisor D such that $K_X + \Delta + D \sim_{\mathbb{Q}} 0$ and $(X, \Delta + D)$ is lc.
(iii) Rational polyhedrality of the Kleiman-Mori cone $\overline{NE}(X)$.

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It is easy to see that (i) implies (ii). In the case where (X, Δ) is a klt pair, the above three properties hold by the Kawamata-Shokurov base point free theorem and the cone theorem (cf. [KMM], [KoM]). Shokurov proved that these three properties hold for surfaces (cf. [S, 2.5. Proposition]).

Among other things, we prove the following:

Theorem 1.4 (=Corollaries 3.3 and 4.5). *Let X be a weak Fano 3-fold with log canonical singularities. Then $-K_X$ is semiample and $\overline{NE}(X)$ is a rational polyhedral cone.*

Theorem 1.5 (=Corollary 3.4 and Theorem 4.4). *Let X be a weak Fano 4-fold with log canonical singularities. Suppose that any lc center of X is at most 1-dimensional. Then $-K_X$ is semiample and $\overline{NE}(X)$ is a rational polyhedral cone.*

On the other hand, the above three properties do not hold for d-dimensional log canonical weak *log* Fano pairs in general, where $d \geq 3$. Indeed, we give the following examples of plt weak log Fano pairs whose anti-log canonical divisors are not semiample in Section 5 (in particular, such examples of 3-dimensional weak log Fano plt pairs show the main results of [Kar1] and [Kar2] do not hold). It is well known that there exists a $(d-1)$-dimensional smooth projective variety S such that $-K_S$ is nef and is not semiample. Let X_0 be the cone over S with respect to some projectively normal embedding $S \subset \mathbb{P}^N$. We take the blow-up X of X_0 at its vertex. Let E be the exceptional divisor of the blow-up. Then the pair (X, E) is a weak log Fano plt pair such that $-(K_X + E)$ is not semiample. Moreover we give an example of a log canonical weak log Fano pair without $\mathbb{Q}$-complements and an example whose Kleiman-Mori cone is not polyhedral.

We now outline the proof of semiampleness of $-K_X$ as in Theorem 1.4. First, we take a birational morphism $\varphi : Y \to X$ such that $\varphi^*(K_X) = K_Y + S$, (Y, S) is dlt and S is reduced. We set $C := \varphi(S)$, which is the union of lc centers of X. By an argument in the proof of the Kawamata-Shokurov base point free theorem (Lemma 2.11), it is sufficient to prove that $-(K_Y + S)|_S$ is semiample. Moreover we have only to prove that $-K_X|_C$ is semiample by the formula $K_X|_C = (\varphi|_S)^*((K_Y + S)|_S)$.

It is not difficult to see semiampleness of the restriction of $-K_X$ on any lc center of X. The main difficulty is how to extend semiampleness to C from each 1-dimensional irreducible component C_i of C since the configuration of C_i's may be complicated. The key to overcome this difficulty is the abundance theorem for 2-dimensional semi-divisorial log terminal pairs ([AFKM]). We decompose $C = C' \cup C''$, where

$$\Sigma := \{i \mid -K_X|_{C_i} \equiv 0\}, \ C' := \bigcup_{i \in \Sigma} C_i, \text{ and } C'' := \bigcup_{i \notin \Sigma} C_i.$$

Let S' be the union of the irreducible components of S whose image on X is contained in C'. We define the boundary $\mathrm{Diff}_{S'}(S)$ on S' by the formula $K_Y + S|_{S'} = K_{S'} + \mathrm{Diff}_{S'}(S)$. The pair $(S', \mathrm{Diff}_{S'}(S))$ is known to be semi-divisorial log terminal pair (sdlt, for short). Applying the abundance theorem to the pair $(S', \mathrm{Diff}_{S'}(S))$, we see that $K_{S'} + \mathrm{Diff}_{S'}(S)$ is $\mathbb{Q}$-linearly trivial, namely, there is a non-zero integer m_1 such that $-m_1(K_Y + S)|_{S'} = -m_1(K_{S'} + \mathrm{Diff}_{S'}(S)) \sim 0$. This shows that

$-m_1K_X|_{C'} \sim 0$. On the other hand, since $-K_X|_{C''}$ is ample, we can take enough sections of $H^0(C'', -m_2K_X|_{C''})$ for a sufficiently large and divisible m_2 (Lemma 2.16). Thus, we can find enough sections of $H^0(C, -mK_X|_C)$ for a sufficiently large and divisible m, and can conclude that $-K_X|_C$ is semiample.

To generalize this theorem to higher dimensional weak log Fano pairs, let us recall the following conjectures:

Conjecture 1.6 (Abundance conjecture in a special case)**.** *Let (X, Δ) be a d-dimensional projective sdlt pair whose $K_X + \Delta$ is numerically trivial. Then $K_X + \Delta$ is $\mathbb{Q}$-linearly trivial, i.e., there exists an $n \in \mathbb{N}$ such that $n(K_X + \Delta) \sim 0$.*

The abundance conjecture is one of the most famous conjecture in the minimal model program. This conjecture is true when $d \leq 3$ by the works of Fujita, Kawamata, Miyaoka, Abramovich, Fong, Kollár, M^{c}Kernan, Keel, Matsuki, and Fujino.

By the same way as in the 3-dimensional case, we see the following theorem:

Theorem 1.7 (=Theorem 3.1)**.** *Assume that* Conjecture 1.6 *in dimension $d-1$ holds. Let (X, Δ) be a d-dimensional log canonical weak log Fano pair. Suppose that $M(X, \Delta) \leq 1$, where*

$$M(X, \Delta) := \max\{\dim P |\ P \textit{ is a lc center of } (X, \Delta)\}.$$

Then $-(K_X + \Delta)$ is semiample.

Indeed, semiampleness of $-K_X$ as in Theorem 1.4 is derived from the above theorem since the singular locus of any normal 3-fold is at most 1-dimensional and Conjecture 1.6 for surfaces holds ([AFKM]). We also derive semiampleness of weak Fano 4-folds such that $M(X, 0) \leq 1$ because Conjecture 1.6 for 3-folds holds ([Fj1]). We remark that by Examples 5.2 and 5.3, this condition for the dimension of lc centers is the best possible.

In Section 4, by the cone theorem for normal varieties by Ambro and Fujino (cf. Theorem 4.3), we derive the following:

Theorem 1.8 (=Theorem 4.4)**.** *Let (X, Δ) be a d-dimensional log canonical weak log Fano pair. Suppose that $M(X, \Delta) \leq 1$. Then $\overline{NE}(X)$ is a rational polyhedral cone.*

Note that rational polyhedrality of $\overline{NE}(X)$ as in Theorem 1.4 is a corollary of the above theorem. In Example 5.6, we also see that the Kleiman-Mori cone is not rational polyhedral in general when $M(X, \Delta) \geq 2$.

This paper is based on the minimal model theory for log canonical pairs developed by Ambro and Fujino ([A1], [A2], [A3], [Fj5], [Fj6], [Fj7]).

We will make use of the standard notation and definitions as in [KoM].

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2. Preliminaries and Lemmas

In this section, we introduce notation and some lemmas for the proof of Theorem 1.7 (=Theorem 3.1).

Definition 2.1. For a $\mathbb{Q}$-Weil divisor $D=\sum_{j=1}^{r} d_jD_j$ such that D_j is a prime divisor for every j and $D_i \neq D_j$ for $i \neq j$, we define the *round-up* $\ulcorner D \urcorner = \sum_{j=1}^{r}\ulcorner d_j \urcorner D_j$ (resp. the *round-down* $\llcorner D \lrcorner = \sum_{j=1}^{r}\llcorner d_j \lrcorner D_j$), where for every real number x, $\ulcorner x \urcorner$ (resp. $\llcorner x \lrcorner$) is the integer defined by $x \leq \ulcorner x \urcorner < x+1$ (resp. $x-1 < \llcorner x \lrcorner \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \llcorner D \lrcorner$. We define

$$D^{=1} = \sum_{d_j=1} D_j, \quad D^{\leq 1} = \sum_{d_j \leq 1} d_jD_j,$$

$$D^{<1} = \sum_{d_j<1} d_jD_j, \quad \text{and} \quad D^{>1} = \sum_{d_j>1} d_jD_j.$$

We call D a *boundary* $\mathbb{Q}$-divisor if $0 \leq d_j \leq 1$ for every j.

Definition 2.2 (Stratum)**.** Let (X,Δ) be an lc pair. A *stratum* of (X,Δ) denotes X itself or an lc center of (X,Δ).

The following theorem is very important as a generalization of vanishing theorems (cf. [A2, Theorem 3.1], [Fj5, Theorem 2.2], [Fj6, Theorem 2.38], [Fj7, Theorem 6.3]).

Theorem 2.3 (Torsion-freeness theorem)**.** *Let Y be a smooth variety and B a boundary $\mathbb{R}$-divisor such that* Supp$B$ *is simple normal crossing. Let $f: Y \to X$ be a projective morphism and L a Cartier divisor on Y such that $H \sim_{\mathbb{R}} L-(K_Y+B)$ is f-semiample. Then every associated prime of $R^qf_*\mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of (Y,B) for any non-negative integer q.*

The following theorem is proved by Fujino ([Fj7, Theorem 10.5]). We include the proof for the reader's convenience.

Theorem 2.4. *Let X be a normal quasi-projective variety and Δ an effective $\mathbb{Q}$-divisor on X such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. Suppose that (X,Δ) is lc. Then there exists a projective birational morphism $\varphi: Y \to X$ from a normal quasi-projective variety with the following properties:*

(i) *Y is $\mathbb{Q}$-factorial,*
(ii) *$a(E,X,\Delta)=-1$ for every φ-exceptional divisor E on Y,*
(iii) *for*

$$\Gamma = \varphi_*^{-1}\Delta + \sum_{E:\varphi\text{-exceptional}} E,$$

it holds that (Y,Γ) is dlt and $K_Y+\Gamma=\varphi^*(K_X+\Delta)$, and

(iv) *Let $\{C_i\}$ be any set of lc centers of (X,Δ). Let $W=\bigcup C_i$ with a reduced structure and S the union of the irreducible components of $\llcorner \Gamma \lrcorner$ which are mapped into W by φ. Then $(\varphi|_S)_*\mathcal{O}_S \simeq \mathcal{O}_W$.*

Proof. Let $\pi : V \to X$ be a resolution such that

(1) $\pi^{-1}(C)$ is a simple normal crossing divisor on V for every lc center C of (X, Δ), and
(2) $\pi_*^{-1}\Delta \cup \mathrm{Exc}(\pi) \cup \pi^{-1}(\mathrm{Nklt}(X, \Delta))$ has a simple normal crossing support, where $\mathrm{Exc}(\pi)$ is the exceptional set of π and $\mathrm{Nklt}(X, \Delta)$ is the union of lc centers of (X, Δ).

By Hironaka's resolution theorem, we can assume that π is a composite of blow-ups with centers of codimension at least two. Then there exists an effective π-exceptional Cartier divisor B on V such that $-B$ is π-ample. We put

$$F = \sum_{\substack{a(E,X,\Delta)>-1,\\ E:\pi\text{-exceptional}}} E \text{ and } G = \sum_{a(E,X,\Delta)=-1} E.$$

Let H be a sufficiently ample Cartier divisor on X such that $-B + \pi^*H$ is ample. We choose $0 < \varepsilon \ll 1$ such that $\varepsilon G - B + \pi^*(H)$ is ample. Since $-B + \pi^*(H)$ and $\varepsilon G - B + \pi^*(H)$ are ample, we can take effective $\mathbb{Q}$-divisors H_1 and H_2 on V with small coefficients such that $G + F + \pi_*^{-1}\Delta + H_1 + H_2$ has a simple normal crossing support and that $-B + \pi^*H \sim_{\mathbb{Q}} H_1$, $\varepsilon G - B + \pi^*(H) \sim_{\mathbb{Q}} H_2$. We take $0 < \nu, \mu \ll 1$ such that every divisor in F has a negative coefficient in

$$M := \Gamma_V - G - (1-\nu)F - \pi_*^{-1}\Delta^{<1} + \mu B,$$

where Γ_V is a $\mathbb{Q}$-divisor on V such that $K_V + \Gamma_V = \pi^*(K_X + \Delta)$. Now we construct a log minimal model of $(V, G + (1-\nu)F + \pi_*^{-1}\Delta^{<1} + \mu H_1)$ over X. Since

$$G + (1-\nu)F + \mu H_1 \sim_{\mathbb{Q}} (1-\varepsilon\mu)G + (1-\nu)F + \mu H_2,$$

it is sufficient to construct a log minimial model of $(V, (1-\varepsilon\mu)G + (1-\nu)F + \pi_*^{-1}\Delta^{<1} + \mu H_2)$ over X. Because $(V, (1-\varepsilon\mu)G + (1-\nu)F + \pi_*^{-1}\Delta^{<1} + \mu H_2)$ is klt, we can get a log minimal model $\varphi : Y \to X$ of $(V, (1-\varepsilon\mu)G + (1-\nu)F + \pi_*^{-1}\Delta^{<1} + \mu H_2)$ over X by [BCHM, Theorem 1.2].

We show this Y satisfies the conditions of the theorem. For any divisor D on V (appearing above), let D' denote its strict transform on Y. We see the following claim:

Claim 2.5. $F' = 0$.

Proof of Claim 2.5. By the above construction,

$$N := K_Y + G' + (1-\nu)F' + \varphi_*^{-1}\Delta^{<1} + \mu H_1'$$

is φ-nef. Then

$$-M' \quad \sim_{\mathbb{Q},\varphi} \quad N - (K_Y + \Gamma_Y)$$

since $(\pi^*H)' = \varphi^*H$, hence it is φ-nef. Since $\varphi_*M' = 0$, we see that M' is effective by the negativity lemma (cf. [KoM, Lemma 3.39]). Since every divisor in F has a negative coefficient in M, F is contracted on Y. We finish the proof of Claim 2.5. □

From Claim 2.5, the discrepancy of every φ-exceptional divisor is equal to -1. We see that Y satisfies the condition (ii). By the above construction, (Y,Γ) is a $\mathbb{Q}$-factorial dlt pair since so is $(Y, G' + \varphi_*^{-1}\Delta^{<1} + \mu H_1)$. We see the condition (i). Because the support of $K_Y + \Gamma - \varphi^*(K_X + \Delta)$ coincide with F', we see the condition (iii).

Now, we show that Y and φ satisfy the condition (iv). Since we get Y by the log minimal model program over X with scaling of some effective divisor with respect to $K_V + G + (1-\nu)F + \pi_*^{-1}\Delta^{<1} + \mu H_1$ (cf. [BCHM]), we see that the rational map $f : V \dashrightarrow Y$ is a composition of $(K_V + G + (1-\nu)F + \pi_*^{-1}\Delta^{<1} + \mu H_1)$-negative divisorial contractions and log flips. Let Σ be an lc center of (Y,Γ). Then it is also an lc center of $(Y, \Gamma + \mu H_1')$. By the negativity lemma, $f : V \dashrightarrow Y$ is an isomorphism around the generic point of Σ. Therefore, if $\varphi(\Sigma) \subseteq W$, then $\Sigma \subseteq S$ by the conditions (1) and (2) for $\pi : V \to X$. This means that no lc centers of $(Y, \Gamma - S)$ are mapped into W by φ. Let $g : Z \to Y$ be a resolution such that

(a) $\operatorname{Supp}\Gamma_Z$ is a simple normal crossing divisor, where Γ_Z is defined by $K_Z + \Gamma_Z = g^*(K_Y + \Gamma)$, and
(b) g is an isomorphism over the generic point of any lc center of (Y,Γ).

Let S_Z be the strict transform of S on Z. We consider the following short exact sequence

$$(*) \qquad \begin{aligned} 0 \to \mathcal{O}_Z(\ulcorner -(\Gamma_Z^{<1})\urcorner - S_Z) &\to \mathcal{O}_Z(\ulcorner -(\Gamma_Z^{<1})\urcorner) \\ &\to \mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner) \to 0. \end{aligned}$$

We note that

$$\ulcorner -(\Gamma_Z^{<1})\urcorner - S_Z - (K_Z + \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z) \sim_{\mathbb{Q}} -h^*(K_X + \Delta),$$

where $h = \varphi \circ g$. Then we obtain

$$\begin{aligned} 0 \to h_*\mathcal{O}_Z(\ulcorner -(\Gamma_Z^{<1})\urcorner - S_Z) \to h_*\mathcal{O}_Z(\ulcorner -(\Gamma_Z^{<1})\urcorner) \to h_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner) \\ \xrightarrow{\delta} R^1h_*\mathcal{O}_Z(\ulcorner -(\Gamma_Z^{<1})\urcorner - S_Z) \to \cdots . \end{aligned}$$

We claim the following:

Claim 2.6. *δ is a zero map.*

Proof of Claim 2.6. Let Σ be an lc center of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$. Then Σ is some intersection of components of $\Gamma_Z^{=1} - S_Z$. By the conditions (a) and (b), $\Gamma_Z^{=1} - S_Z$ is the strict transform of $\llcorner \Gamma \lrcorner - S$. By this, the image of Σ by g is some intersection of components of $\llcorner \Gamma \lrcorner - S$. In particular, $g(\Sigma)$ is an lc center of $(Y, \Gamma - S)$. Thus no lc centers of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$ are mapped into W by h. Assume by contradiction that δ is not zero. Then there exists a section $s \in H^0(U, h_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner))$ for some non-empty open set $U \subseteq X$ such that $\delta(s) \neq 0$. Since $\operatorname{Supp}\delta(s) \neq \emptyset$, we can take an associated prime $x \in \operatorname{Supp}\delta(s)$. We see that $x \in W$ since $\operatorname{Supp}(h_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner))$ is contained in W. By Theorem 2.3, x is the generic point of the h-image of some stratum of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$. Since h is a birational morphism, x is the generic point of the h-image of some lc center of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$. Because no lc centers of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$ are mapped into W by h, it holds that $x \notin W$. But this contradicts the way of taking x.

□

Thus, we obtain

$$0 \to \mathcal{I}_W \to \mathcal{O}_X \to h_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner) \to 0,$$

where $\mathcal{I}_W$ is the defining ideal sheaf of W since $\ulcorner -(\Gamma_Z^{<1})\urcorner$ is effective and h-exceptional. This implies that $\mathcal{O}_W \simeq h_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner)$. By applying g_* to $(*)$, we obtain

$$0 \to \mathcal{I}_S \to \mathcal{O}_Y \to g_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner) \to 0,$$

where $\mathcal{I}_S$ is the defining ideal sheaf of S since $\ulcorner -(\Gamma_Z^{<1})\urcorner$ is effective and g-exceptional. We note that

$$R^1g_*\mathcal{O}_Z(\ulcorner -(\Gamma_Z^{<1})\urcorner - S_Z) = 0$$

by Theorem 2.3 since g is an isomorphism at the generic point of any stratum of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$. Thus, $\mathcal{O}_W \simeq h_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner) \simeq \varphi_*g_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner) \simeq \varphi_*\mathcal{O}_S$. We finish the proof of Theorem 2.4. □

Definition 2.7. Let X be a normal variety and D a $\mathbb{Q}$-Weil divisor. We define that

$$R(X, D) = \bigoplus_{m=0}^{\infty} H^0(X, \llcorner mD \lrcorner).$$

Definition 2.8 (semi-divisorial log terminal, cf. [Fj1])**.** Let X be a reduced S_2-scheme. We assume that it is pure d-dimensional and is normal crossing in codimension 1. Let Δ be an effective $\mathbb{Q}$-Weil divisor on X such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier.

Let $X = \bigcup X_i$ be the decomposition into irreducible components, and $\nu : X' := \coprod X'_i \to X = \bigcup X_i$ the normalization. Define the $\mathbb{Q}$-divisor Θ on X' by $K_{X'} + \Theta := \nu^*(K_X + \Delta)$ and set $\Theta_i := \Theta|_{X'_i}$.

We say that (X, Δ) is *semi-divisorial log terminal* (for short, *sdlt*) if X_i is normal, that is, X'_i is isomorphic to X_i, and (X'_i, Θ_i) is dlt for every i.

Definition and Lemma 2.9 (Different, cf. [C])**.** Let (Y, Γ) be a dlt pair and S a union of some components of $\llcorner \Gamma \lrcorner$. Then there exists an effective $\mathbb{Q}$-divisor $\mathrm{Diff}_S(\Gamma)$ on S such that $(K_Y + \Gamma)|_S \sim_{\mathbb{Q}} K_S + \mathrm{Diff}_S(\Gamma)$. The effective $\mathbb{Q}$-divisor $\mathrm{Diff}_S(\Gamma)$ is called the *different* of Γ. Moreover it holds that $(S, \mathrm{Diff}_S(\Gamma))$ is sdlt.

The following proposition is [Fk2, Proposition 2] (for the proof, see [Fk1, Proof of Theorem 3] and [Kaw, Lemma 3]).

Proposition 2.10. *Let (X, Δ) be a proper dlt pair and L a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and big for some $a \in \mathbb{N}$. If $\mathrm{Bs}|mL| \cap \llcorner \Delta \lrcorner = \emptyset$ for every $m \gg 0$, then $|mL|$ is base point free for every $m \gg 0$, where $\mathrm{Bs}|mL|$ is the base locus of $|mL|$.*

By this proposition, we derive the following lemma:

Lemma 2.11. *Let (Y, Γ) be a $\mathbb{Q}$-factorial weak log Fano dlt pair. Suppose that $-(K_S + \Gamma_S)$ is semiample, where $S := \llcorner \Gamma \lrcorner$ and $\Gamma_S := \mathrm{Diff}_S(\Gamma)$. Then $-(K_Y + \Gamma)$ is semiample.*

Proof. We consider the exact sequence

$$0 \to \mathcal{O}_Y(-m(K_Y+\Gamma)-S) \to \mathcal{O}_Y(-m(K_Y+\Gamma)) \to \\ \to \mathcal{O}_S(-m(K_Y+\Gamma)|_S) \to 0$$

for $m \gg 0$. By the Kawamata-Viehweg vanishing theorem (cf. [KMM, Theorem 1-2-5.], [KoM, Theorem 2.70]), we have

$$H^1(Y, \mathcal{O}_Y(-m(K_Y+\Gamma)-S)) = \\ = H^1(Y, \mathcal{O}_Y(K_Y+\Gamma-S-(m+1)(K_X+\Gamma))) = 0,$$

since the pair $(Y, \Gamma - S)$ is klt and $-(K_Y+\Gamma)$ is nef and big. Thus, we get the exact sequence

$$H^0(Y, \mathcal{O}_Y(-m(K_Y+\Gamma)) \to H^0(S, \mathcal{O}_S(-m(K_Y+\Gamma)|_S)) \to 0.$$

Therefore, we see that $\mathrm{Bs}|-m(K_Y+\Gamma)| \cap S = \emptyset$ for $m \gg 0$ since $-(K_S+\Delta_S)$ is semiample. Applying Proposition 2.10, we conclude that $-(K_Y+\Gamma)$ is semiample. □

Definition 2.12. (cf. [GT, 1.1. Definition], [KoS, Definition 7.1]) Suppose that R is a reduced excellent ring and $R \subseteq S$ is a reduced R-algebra which is finite as an R-module. We say that the extension $i : R \hookrightarrow S$ is *subintegral* if one of the following equivalent conditions holds:

(a) $(S \bigotimes_R k(\mathfrak{p}))_{\mathrm{red}} = k(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.
(b) the induced map on the spectra is bijective and i induces trivial residue field extensions.

Definition 2.13. [KoS, Definition 7.2] Suppose that R is a reduced excellent ring. We say that R is *seminormal* if every subintegral extension $R \hookrightarrow S$ is an isomorphism.

A scheme X is called *seminormal at $q \in X$* if the local ring at q is seminormal. If X is seminormal at every point, we say that X is *seminormal*.

Proposition 2.14. [GT, 5.3. Corollary] *Let $(R, \mathfrak{m})$ be a local excellent ring. Then R is seminormal if and only if $\widehat{R}$ is seminormal, where $\widehat{R}$ is $\mathfrak{m}$-adic completion of R.*

Proposition 2.15. (*cf.* [Ko, 7.2.2.1], [KoS, Remark 7.6]) *Let C be a pure 1-dimensional proper reduced scheme of finite type over $\mathbb{C}$, and $q \in C$ a closed point. Then C is seminormal at q if and only if $\widehat{\mathcal{O}}_{C,q}$ satisfies that*

(i) $\widehat{\mathcal{O}}_{C,q} \simeq \mathbb{C}[[X]]$, *or*
(ii) $\widehat{\mathcal{O}}_{C,q} \simeq \mathbb{C}[[X_1, X_2, \cdots, X_r]]/\langle X_iX_j | 1 \le i \ne j \le r\rangle$ *for some $r \ge 2$, i.e., $q \in C$ is isomorphic to the coordinate axies in $\mathbb{C}^r$ at the origin as a formal germs.*

Lemma 2.16. *Let $C = C_1 \cup C_2$ be a pure 1-dimensional proper seminormal reduced scheme of finite type over $\mathbb{C}$, where C_1 and C_2 are pure 1-dimensional reduced closed subschemes. Let D be a $\mathbb{Q}$-Cartier divisor on C. Suppose that D_1 is $\mathbb{Q}$-linearly trivial and D_2 is ample, where $D_i := D|_{C_i}$. Then D is semiample.*

Proof. Let $C_1 \cap C_2 = \{p_1, \dots, p_r\}$. We take $m \gg 0$ which satisfies the following:

(i) $mD_1 \sim 0$,

(ii) $\mathcal{O}_{C_2}(mD_2) \otimes (\bigcap_{k \neq l} \mathfrak{m}_{p_k})$ is generated by global sections for all $l \in \{1, \dots, r\}$, and

(iii) $\mathcal{O}_{C_2}(mD_2) \otimes (\bigcap_{k} \mathfrak{m}_{p_k})$ is generated by global sections,

where $\mathfrak{m}_{p_k}$ is the ideal sheaf of p_k on C_2. We choose a nowhere vanishing section $s \in H^0(C_1, mD_1)$. By (ii), we can take a section $t_l \in H^0(C_2, mD_2)$ which does not vanish at p_l but vanishes at all the p_k ($k \in \{1, \dots, r\}, k \neq l$) for each $l \in \{1, \dots, r\}$. By multiplying suitable nonzero constants to t_l, we may assume that $t_l|_{p_l} = s|_{p_l}$. We set $t := \sum_l t_l \in H^0(C_2, mD_2)$. Since C is seminormal, Proposition 2.15 implies that $\mathcal{O}_{C_1 \cap C_2} \simeq \bigoplus_{l=1}^{r} \mathbb{C}(p_l)$, where $\mathbb{C}(p_l)$ is the skyscraper sheaf $\mathbb{C}$ sitting at p_l, by computations on $\widehat{\mathcal{O}}_{C,p_l}$. Thus we get the following exact sequence:

$$0 \to \mathcal{O}_C(mD) \to \mathcal{O}_{C_1}(mD_1) \oplus \mathcal{O}_{C_2}(mD_2) \to \bigoplus_{l=1}^{r} \mathbb{C}(p_l) \to 0,$$

where the third arrow maps (s', s'') to $((s' - s'')|_{p_1}, \dots, (s' - s'')|_{p_r})$. Hence s and t patch together and give a section u of $H^0(C, mD)$.

Let p be any point of C. If $p \in C_1$, then u does not vanish at p. We may assume that $p \in C_2 \setminus C_1$. By (iii), we can take a section $t' \in H^0(C_2, mD_2)$ which does not vanish at p but vanishes at p_l for all $l \in \{1, \dots, r\}$. The zero section $0 \in H^0(C_1, mC_1)$ and t' patch together and give a section u' of $H^0(C, mD)$. By construction, the section u' does not vanish at p. We finish the proof of Lemma 2.16. $\square$

3. On semiampleness for weak Fano varieties

In this section, we prove Theorem 1.7 (=Theorem 3.1). As a corollary, we see that the anti-canonical divisors of weak Fano 3-folds with log canonical singularities are semiample. Moreover we derive semiampleness of the anti-canonical divisors of log canonical weak Fano 4-folds whose lc centers are at most 1-dimensional.

Theorem 3.1. *Assume that* Conjecture 1.6 *in dimension* $d-1$ *holds. Let* (X, Δ) *be a* d*-dimensional log canonical weak log Fano pair. Suppose that* $M(X, \Delta) \leq 1$*, where*

$$M(X, \Delta) := \max\{\dim P |\ P \textit{ is a lc center of } (X, \Delta)\}.$$

Then $-(K_X + \Delta)$ *is semiample.*

Proof. By Theorem 2.4, we take a birational morphism $\varphi : (Y, \Gamma) \to (X, \Delta)$ as in the theorem. We set $S := \llcorner \Gamma \lrcorner$ and $C := \varphi(S)$, where we consider the reduced scheme structures on S and C. We have only to prove that $-(K_S + \Gamma_S) = -(K_Y + \Gamma)|_S$ is semiample from Lemma 2.11. By the formula $(K_Y + \Gamma)|_S \sim_{\mathbb{Q}} (\varphi|_S)^*((K_X + \Delta)|_C)$, it suffices to show that $-(K_X + \Delta)|_C$ is semiample. Arguing on each connected component of C, we may assume that C is connected. Since $M(X, \Delta) \leq 1$, it holds that $\dim C \leq 1$. When $\dim C = 0$, i.e., C is a closed point, then $-(K_X + \Delta)|_C \sim_{\mathbb{Q}} 0$, in particular, is semiample.

When $\dim C = 1$, C is a pure 1-dimensional seminormal scheme by [A3, Theorem

1.1] or [Fj7, Theorem 9.1]. Let $C = \bigcup_{i=1}^{r} C_i$, where C_i is an irreducible component, and let $D := -(K_X + \Delta)|_C$ and $D_i := D|_{C_i}$. We set

$$\Sigma := \{i|\ D_i \equiv 0\},\ C' := \bigcup_{i\in\Sigma} C_i,\ C'' := \bigcup_{i\notin\Sigma} C_i.$$

Let S' be the union of irreducible components of S whose image by φ is contained in C'. We see that $K_{S'} + \Gamma_{S'} \equiv 0$, where $\Gamma_{S'} := \mathrm{Diff}_{S'}(\Gamma)$. Thus it holds that $K_{S'} + \Gamma_{S'} \sim_{\mathbb{Q}} 0$ by applying Conjecture 1.6 to $(S', \Gamma_{S'})$. Since $(\varphi|_{S'})_*\mathcal{O}_{S'} \simeq \mathcal{O}_{C'}$ by the condition (iv) in Theorem 2.4, it holds that $D|_{C'} \sim_{\mathbb{Q}} 0$. We see that $D|_{C''}$ is ample since the restriction of D on any irreducible component of C'' is ample. By Lemma 2.16, we see that $D = -(K_X + \Delta)|_C$ is semiample. We finish the proof of Theorem 3.1. □

Corollary 3.2. *Assume that* Conjecture 1.6 *in dimension* $d-1$ *holds.*
Let (X, Δ) *be a* d*-dimensional log canonical weak log Fano pair. Suppose that* $M(X, \Delta) \leq 1$*. Then* $R(X, -(K_X + \Delta))$ *is a finitely generated algebra over* $\mathbb{C}$*.*

Conjecture 1.6 holds for surfaces and 3-folds by [AFKM] and [Fj1]. Thus we immediately obtain the following corollaries:

Corollary 3.3. *Let* (X, Δ) *be a* 3*-dimensional log canonical weak log Fano pair. Suppose that* $\llcorner \Delta \lrcorner = 0$*. Then* $-(K_X + \Delta)$ *is semiample and* $R(X, -(K_X + \Delta))$ *is a finitely generated algebra over* $\mathbb{C}$*. In particular, if* X *is a weak Fano* 3*-fold with log canonical singularities, then* $-K_X$ *is semiample and* $R(X, -K_X)$ *is a finitely generated algebra over* $\mathbb{C}$*.*

Corollary 3.4. *Let* (X, Δ) *be a* 4*-dimensional log canonical weak log Fano pair. Suppose that* $M(X, \Delta) \leq 1$*. Then* $-(K_X + \Delta)$ *is semiample and* $R(X, -(K_X + \Delta))$ *is a finitely generated algebra over* $\mathbb{C}$*. In particular, if* X *is a log canonical weak Fano* 4*-fold whose lc centers are at most* 1*-dimensional, then* $-K_X$ *is semiample and* $R(X, -K_X)$ *is a finitely generated algebra over* $\mathbb{C}$*.*

Remark 3.5. *When* $M(X, \Delta) \geq 2$*,* $-(K_X+\Delta)$ *is not semiample and* $R(X, -(K_X+\Delta))$ *is not a finitely generated algebra over* $\mathbb{C}$*, in general* (Examples 5.2 *and* 5.3)*.*

Remark 3.6. *Based on* Theorem 3.1*, we expect the following statement:*

> *Let* (X, Δ) *be lc pair and* D *a nef Cartier divisor. Suppose there is a positive number* a *such that* $aD - (K_X + \Delta)$ *is nef and big. If it holds that* $M(X, \Delta) \leq 1$*, then* D *is semiample.*

However, there is a counterexample for this statement due to Zariski (*cf.* [KMM, Remark 3-1-2]*,* [Z]*).*

4. On the Kleiman-Mori cone for weak Fano varieties

In this section, we introduce the cone theorem for normal varieties by Ambro and Fujino and prove polyhedrality of the Kleiman-Mori cone for a log canonical weak Fano variety whose lc centers are at most 1-dimensional. We use the notion of the scheme $\mathrm{Nlc}(X, \Delta)$, whose underlying space is the set of non-log canonical singularities. For the scheme structure on $\mathrm{Nlc}(X,\Delta)$, we refer [Fj7, Section 7] and [Fj4] in detail.

Definition 4.1. ([Fj7, Definition 16.1]) Let X be a normal variety and Δ an effective $\mathbb{Q}$-divisor on X such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. Let $\pi : X \to S$ be a projective morphism. We put

$$\overline{NE}(X/S)_{\mathrm{Nlc}(X,\Delta)} = \mathrm{Im}(\overline{NE}(\mathrm{Nlc}(X,\Delta)/S) \to \overline{NE}(X/S)).$$

Definition 4.2. ([Fj7, Definition 16.2]) An *extremal face* of $\overline{NE}(X/S)$ is a non-zero subcone $F \subset \overline{NE}(X/S)$ such that $z, z' \in F$ and $z+z' \in F$ implies that $z, z' \in F$. Equivalently, $F = \overline{NE}(X/S) \cap H^{\perp}$ for some π-nef $\mathbb{R}$-divisor H, which is called a *supporting function* of F. An *extremal ray* is a one-dimensional extremal face.

(1) An extremal face F is called $(K_X+\Delta)$-*negative* if

$$F \cap \overline{NE}(X/S)_{K_X+\Delta\geq 0} = \{0\}.$$

(2) An extremal face F is called *rational* if we can choose a π-nef $\mathbb{Q}$-divisor H as a support function of F.

(3) An extremal face F is called *relatively ample at* $\mathrm{Nlc}(X,\Delta)$ if

$$F \cap \overline{NE}(X/S)_{\mathrm{Nlc}(X,\Delta)} = \{0\}.$$

Equivalently, $H|_{\mathrm{Nlc}(X,\Delta)}$ is $\pi|_{\mathrm{Nlc}(X,\Delta)}$-ample for every supporting function H of F.

(4) An extremal face F is called *contractible at* $\mathrm{Nlc}(X,\Delta)$ if it has a rational supporting function H such that $H|_{\mathrm{Nlc}(X,\Delta)}$ is $\pi|_{\mathrm{Nlc}(X,\Delta)}$-semiample.

Theorem 4.3. (Cone theorem for normal varieties, [A2, Theorem 5.10], [Fj7, Theorem 16.5]) *Let X be a normal variety, Δ an effective $\mathbb{Q}$-divisor on X such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier, and $\pi : X \to S$ a projective morphism. Then we have the following properties.*

(1) $\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X+\Delta\geq 0} + \overline{NE}(X/S)_{\mathrm{Nlc}(X,\Delta)} + \sum R_j$, *where R_j's are the $(K_X+\Delta)$-negative extremal rays of $\overline{NE}(X/S)$ that are rational and relatively ample at* $\mathrm{Nlc}(X,\Delta)$. *In particular, each R_j is spanned by an integral curve C_j on X such that $\pi(C_j)$ is a point.*

(2) *Let H be a π-ample $\mathbb{Q}$-divisor on X. Then there are only finitely many R_j's included in $(K_X+\Delta+H)_{<0}$. In particular, the R_j's are discrete in the half-space $(K_X+\Delta)_{<0}$.*

(3) *Let F be a $(K_X+\Delta)$-negative extremal face of $\overline{NE}(X/S)$ that is relatively ample at* $\mathrm{Nlc}(X,\Delta)$. *Then F is a rational face. In particular, F is contractible at* $\mathrm{Nlc}(X,\Delta)$.

By the above Theorem, we derive the following theorem:

Theorem 4.4. *Let (X,Δ) be a d-dimensional log canonical weak log Fano pair. Suppose that $M(X,\Delta)\leq 1$. Then $\overline{NE}(X)$ is a rational polyhedral cone.*

Proof. Since $-(K_X+\Delta)$ is nef and big, there exists an effective divisor B satisfies the following: for any sufficiently small rational positive number ε, there exists a general $\mathbb{Q}$-ample divisor A_ε such that

$$-(K_X+\Delta)\sim_{\mathbb{Q}}\varepsilon B+A_\varepsilon.$$

We fix a sufficiently small rational positive number ε and set $A:=A_\varepsilon$. We also take a sufficiently small positive number δ. Thus $\mathrm{Supp}(\mathrm{Nlc}(X,\Delta+\varepsilon B+\delta A))$ is contained in the union of lc centers of (X,Δ) and $-(K_X+\Delta+\varepsilon B+\delta A)$ is ample. By applying Theorem 4.3 to $(X,\Delta+\varepsilon B+\delta A)$, We get

$$\overline{NE}(X)=\overline{NE}(X)_{\mathrm{Nlc}(X,\Delta+\varepsilon B+\delta A)}+\sum_{j=1}^{m}R_j \text{ for some } m.$$

Now we see that $\overline{NE}(X)_{\mathrm{Nlc}(X,\Delta+\varepsilon B+\delta A)}$ is polyhedral since $\dim\mathrm{Nlc}(X,\Delta+\varepsilon B)\le 1$ by the assumption of $M(X,\Delta)\le 1$. We finish the proof of Theorem 4.4.

□

Corollary 4.5. *Let X be a weak Fano 3-fold with log canonical singularities. Then the cone $\overline{NE}(X)$ is rational polyhedral.*

Remark 4.6. *When $M(X,\Delta)\ge 2$, $\overline{NE}(X)$ is not polyhedral in general* (Example 5.6).

5. Examples

In this section, we construct examples of log canonical weak log Fano pairs (X,Δ) such that $-(K_X+\Delta)$ is not semiample, (X,Δ) does not have $\mathbb{Q}$-complements, or $\overline{NE}(X)$ is not polyhedral.

Basic construction 5.1. Let S be a $(d-1)$-dimensional smooth projective variety such that $-K_S$ is nef and $S\subset\mathbb{P}^N$ some projectively normal embedding. Let X_0 be the cone over S and $\phi\colon X\to X_0$ the blow-up at the vertex. Then the linear projection $X_0 \dashrightarrow S$ from the vertex is decomposed as follows:

$$\begin{array}{ccc} & X & \\ {\scriptstyle\phi}\swarrow & & \searrow{\scriptstyle\pi} \\ X_0 & & S. \end{array}$$

This diagram is the restriction of the diagram for the projection $\mathbb{P}^{N+1}\dashrightarrow\mathbb{P}^N$:

$$\begin{array}{ccc} & V:=\mathbb{P}_{\mathbb{P}^N}(\mathcal{O}_{\mathbb{P}^N}\oplus\mathcal{O}_{\mathbb{P}^N}(-1)) & \\ {\scriptstyle\phi_0}\swarrow & & \searrow{\scriptstyle\pi_0} \\ \mathbb{P}^{N+1} & & \mathbb{P}^N. \end{array}$$

Moreover, the ϕ_0-exceptional divisor is the tautological divisor of $\mathcal{O}_{\mathbb{P}^N}\oplus\mathcal{O}_{\mathbb{P}^N}(-1)$. Hence $X\simeq\mathbb{P}_S(\mathcal{O}_S\oplus\mathcal{O}_S(-H))$, where H is a hyperplane section on $S\subset\mathbb{P}^N$, and the ϕ-exceptional divisor E is isomorphic to S and is the tautological divisor of $\mathcal{O}_S\oplus\mathcal{O}_S(-H)$.

By the canonical bundle formula, it holds that

$$K_X = -2E + \pi^*(K_S - H),$$

thus we have

$$-(K_X + E) = \pi^*(-K_S) + \pi^*H + E$$

We see $\pi^*H + E$ is nef and big since $\mathcal{O}_X(\pi^*(H) + E) \simeq \phi^*\mathcal{O}_{X_0}(1)$ and ϕ is birational. Hence $-(K_X + E)$ is nef and big since $\pi^*(-K_S)$ is nef.

The above construction is inspired by that of Hacon and M^{c}Kernan in Lazić's paper (cf. [Lc, Theorem A.6]).

In the following examples, (X, E) is the plt weak log Fano pair given by the above construction.

Example 5.2. This is an example of a d-dimensional plt weak log Fano pair such that the anti-log canonical divisors are not semiample, where $d \geq 3$.
There exists a variety S such that $-K_S$ is nef and is not semiample (e.g. the surface obtained by blowing up $\mathbb{P}^2$ at very general 9 points). We see that $-(K_X + E)$ is not semiample since $-(K_X + E)|_E = -K_E$ is not semiample. In particular, $R(X, -(K_X + E))$ is not a finitely generated algebra over $\mathbb{C}$ by $-(K_X + \Delta)$ is nef and big.

Example 5.3. This is an example of a log canonical weak Fano variety such that the anti-canonical divisor is not semiample.
Let T be a k-dimensional smooth projective variety whose $-K_T$ is nef and A a $(d - k - 1)$-dimensional smooth projective manifold with $K_A \sim_{\mathbb{Q}} 0$, where d and k are integers satisfying $d - 1 \geq k \geq 0$. We set $S = A \times T$. Let $p_T : S \to T$ be the canonical projection. We see that $K_S = p_T^*(K_T)$. Let A_p be the fiber of p_T at a point $p \in T$, and $\varphi : X \to Y$ the birational morphism with respect to $|\phi^*(\mathcal{O}_{X_0}(1)) \otimes \pi^* p_T^* \mathcal{O}_T(H_T)|$, where H_T is some very ample divisor on T. We claim the following:

Claim 5.4. *It holds that:*

(i) *Y is a projective variety with log canonical singularities.*
(ii) $\mathrm{Exc}(\varphi) = E$ *and any exceptional curve of φ is contained in some A_p.*
(iii) $\varphi^*K_Y = K_X + E$.
(iv) $\varphi(E) = T$ *and* $(\varphi|_E)^*K_T = K_E$.

Proof of Claim 5.4. We see (ii) easily. Since $K_X + E$ is φ-trivial and $-E|_E$ is ample, we see (iii). (i) follows from (iii). By (iii), $\varphi(E)$ is a lc center. By $(\phi^*(\mathcal{O}_{X_0}(1)) \otimes \pi^* p_T^* \mathcal{O}_T(H_T))|_E \simeq p_T^* \mathcal{O}_T(H_T)$, it holds that $\varphi|_E = p_T$. Thus (iv) follows. □

If $-K_T$ is not semiample, then $-K_Y$ is not semiample and $k \geq 2$. Thus we see that Y is a log canonical weak Fano variety with $M(Y, 0) = k$ and $-K_Y$ is not semiample. In particular, $R(X, -K_X)$ is not a finitely generated algebra over $\mathbb{C}$ by $-K_X$ is nef and big (cf. [Lf, Theorem 2.3.15]).

Example 5.5. We construct an example of a weak log Fano plt pair without $\mathbb{Q}$-complements.

Let S be the $\mathbb{P}^1$-bundle over an elliptic curve with respect to a non-split vector bundle of degree 0 and rank 2. Then $-K_S$ is nef and S does not have $\mathbb{Q}$-complements (cf. [S, 1.1. Example]). Thus (X, E) does not have $\mathbb{Q}$-complements by the adjunction formula $-(K_X + E)|_E = -K_E$.

Example 5.6. We construct an example of a weak log Fano plt pair whose Kleiman-Mori cone is not polyhedral. Let S be the surface obtained by blowing up $\mathbb{P}^2$ at very general 9 points. It is well known that S has infinitely many (-1)-curves $\{C_i\}$.
Then we see that the Kleiman-Mori cone $\overline{NE}(X)$ is not polyhedral. Indeed, we have the following claim:

Claim 5.7. $\mathbb{R}_{\geq 0}[C_i] \subseteq \overline{NE}(X)$ *is an extremal ray with* $(K_X + E).C_i = -1$*. Moreover, it holds that* $\mathbb{R}_{\geq 0}[C_i] \neq \mathbb{R}_{\geq 0}[C_j]$ $(i \neq j)$*.*

Proof of Claim 5.7. We take a semiample line bundle L_i on S such that L_i such that L_i satisfies $L_i.C_i = 0$ and $L_i.G > 0$ for any pseudoeffective curve $[G] \in \overline{NE}(S)$ such that $[G] \notin \mathbb{R}_{\geq 0}[C_i]$. We identify E with S. Let $\mathcal{L}_i$ be a pullback of L_i by π and $\mathcal{F}_i := \phi^*(\mathcal{O}_{X_0}(1)) \otimes \mathcal{L}_i$. We show that $\mathbb{R}_{\geq 0}[C_i] \subseteq \overline{NE}(X)$ is an extremal ray. Since $(K_X + E)|_E \sim K_E$, it holds that $(K_X + E).C_i = -1$. By the cone theorem for dlt pairs, there exist finitely many $(K_X + E)$-negative extremal rays R_k such that $[C_i] - [D] \in \sum R_k$ for some $[D] \in \overline{NE}(X)_{K_X+E=0}$. It holds that $\mathcal{F}_i.D = \mathcal{F}_i.R_k = 0$ for all k since $\mathcal{F}_i.C_i = 0$ and $\mathcal{F}_i$ is a nef line bundle. We see that, if an effective 1-cycle C on X satisfies $\mathcal{F}_i.C = 0$, then $C = \alpha C_i$ for some $\alpha \geq 0$ by the construction of $\mathcal{F}_i$. Thus, any generator of R_k is equal to $\alpha_k C_i$ for some $\alpha_k \geq 0$. Hence $\mathbb{R}_{\geq 0}[C_i] \subseteq \overline{NE}(X)$ is an extremal ray. It is clear to see that $\mathbb{R}_{\geq 0}[C_i] \neq \mathbb{R}_{\geq 0}[C_j]$. Thus the claim holds. □

References

[AFKM] D. Abramovich, L. L. Y. Fong, J. Kollár and J. M^cKernan, *Semi log canonical surface*, Flip and Abundance for algebraic threefolds, Astérisque 211 (1992), 139–154.

[A1] F. Ambro, *The locus of log canonical singularities*, math.AG/9806067.

[A2] ———, *Quasi-log varieties*, Tr. Mat. Inst. Steklova **240** (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 220–239; translation in Proc. Steklov Inst. Math. 2003, no. 1 (240), 214–233.

[A3] ———, *Basic properties of log canonical centers*, math.AG/0611205.

[BCHM] C. Birkar, P. Cascini, C. D. Hacon and J. M^cKernan, *Existence of minimal models for varieties of log general type*, arXiv:0610.0203v2, to appear in J. Amer. Math. Soc.

[C] A. Corti, *Adjunction of log divisors*, Flip and Abundance for algebraic threefolds, Astérisque 211 (1992), 171–182.

[Fk1] S. Fukuda, *On base point free theorem*, Kodai Math. J. **19** (1996), 191–199.

[Fk2] ———, *A base point free theorem of Reid type*, J. Math. Sci. Univ. Tokyo **4** (1997), 621–625.

[Fj1] O. Fujino, *Abundance Theorem for semi log canonical threefolds*, Duke Math. J. **102** (2000), no. 3, 513–532.

[Fj2] ———, *Base point free theorem of Reid-Fukuda type*, J. Math. Sci. Univ. Tokyo **7** (2000), no. 1, 1–5.

[Fj3] ———, *What is log terminal?*, Flips for 3-folds and 4-folds, Oxford Lecture Series in Mathematics and its Applications, vol.35, Oxford University Press, 2007, 49–62.

[Fj4] ———, *Theory of non-lc ideal sheaves—basic properties—*, arXiv:0801.2198, to appear in Kyoto Journal of Mathematics.

[Fj5] ———, *Introduction to the theory of quasi-log varieties*, arXiv:0711.1203.

[Fj6] ———, *Introduction to the log minimal model program for log canonical pairs*, arXiv:0907.1506.

[Fj7] ———, *Fundamental theorems for the log minimal model program*, arXiv:0909.4445.

[GT] S. Greco and C. Travero, *On seminormal schemes*, Comp. Math. **40** (1980), no. 3, 325–365.

[Kar1] I. V. Karzhemanov, *Semiampleness theorem for weak log Fano varieties*, Russ. Acad. Sci. Sb. Math. **197** (2006), 57–64.

[Kar2] ———, *Base point free theorem for weak log Fano threefolds*, arXiv:0906.0553.

[Kaw] Y. Kawamata, *Log canonical models of algebraic* 3*-folds*, Internat. J. Math. **3** (1992), 351–357.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, Algebraic Geometry, Sendai 1985, Adv. Stud. Pure Math. **10** (1987), Kinokuniya and North-Holland, 283–360.

[Ko] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics.

[KoM] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math. 134 (1998).

[KoS] S. J. Kovács and K. Schwede, *Hodge theory meets the minimal model program: a survey of log canonical and Du Bois singularities*, arXiv:0909.0993.

[Lf] R. Lazarsfeld, *Positivity in algebraic geometry I*, Ergeb. Math. Grenzgeb. vol. 48, Springer-Verlag, Berlin, 2004.

[Lc] V. Lazić, *Adjoint rings are finitely generated*, arXiv:0905.2707.

[P] Y. G. Prokhorov, *Lectures on complements on log surfaces*, MSJ Memoirs 10 (2001).

[S] V. V. Shokurov, *Complements on surfaces*, J. Math. Sci. **107** (2000), no. 2, 3876–3932.

[Z] O. Zariski, *The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface*, Ann. of Math. **76** (1962), 560–615.

Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan.

E-mail address: gongyo@ms.u-tokyo.ac.jp