ON VARIETIES ADMITTING RATIONALLY CONNECTED AMPLE DIVISORS

YOSHINORI GONGYO

1. Introduction

In this paper, we work over the complex number field $\mathbb{C}$. The following theorems are very well known:

Theorem 1.1. *Let X be a projective manifold and A submanifold of X. Suppose that the normal bundle $\mathcal{N}_{A/X}$ is nef. Then X is uniruled if A is so.*

Theorem 1.2. *Let X be a projective manifold and A submanifold of X. Suppose that the normal bundle $\mathcal{N}_{A/X}$ is ample. Then X is rationally connected if A is so.*

The above theorems are proved by using the deformation of rational curves (cf. [AK], [Ko, Chapter IV]). In this paper we consider these theorems when X is singular and A is codimension 1.

We prove the following theorem:

Theorem 1.3. *Let X be a $\mathbb{Q}$-Gorenstein normal projective variety, A a semi-ample and big Cartier divisor on X such that A is a uniruled variety with only canonical singularities. Suppose that X has $\mathbb{Q}$-factorial and Cohen–Macaulay around A. Then X is uniruled.*

Theorem 1.4. *Let X be a $\mathbb{Q}$-factorial Cohen–Macaulay normal projective variety and A an ample Cartier divisor on X such that A is a rationally connected variety with only canonical singularities. Then X is rationally connected.*

Theorem 1.3 has concern with [Kop] and [PSS] which study about the relation of Kodaira dimensions of X and A. It is difficult to show that X is uniruled if $\kappa(X) = -\infty$. So it dose not seem to show Theorem 1.3 directly from [Kop] and [PSS]. On the other hand, [P] studied about the uniruledness of X. In this paper, Peternell generalized Theorem 1.1 in

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the case where X, A have only canonical singularities, $\mathrm{codim}_A(X_{\mathrm{Sing}} \cap A) \geq 0$, A is not of general type and $\mathcal{N}_{A/X}$ is ample. However our proof is quite different from these papers.

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2. Preliminaries

In this section, we introduce notations.

Definition 2.1. Let X be a normal variety and Δ an effective $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor. Let $\varphi : Y \to X$ be a log resolution of (X, Δ). We set

$$K_Y = \varphi^*(K_X + \Delta) + \sum a_i E_i,$$

where E_i is a prime divisor. The pair (X, Δ) is called *kawamata log terminal* (*klt*, for short) if $a_i > -1$ for all i. Moreover, we call X a *log terminal* variety when $(X, 0)$ is *klt*. In particular we say that X has only canonical singularities if it holds for $(X, 0)$ that $a_i > 0$ for all i.

Definition 2.2. Let X be a normal and proper variety. A dominant rational map $\pi : X \dashrightarrow W$ is called a *rationally chain connected fibration* (*RCC-fibration*, for short) if there exist open sets $X_0 \subseteq X$ and $Z_0 \subseteq Z$ such that $\pi_0 :=$ the restricton of π on X_0 satisfies the following;

(1) π_0 is a proper morphism from X_0 to Z_0.
(2) every fiber of π is connected rationally chain connected .

In paticular, RCC-fibration $\pi : X \dashrightarrow W$ is called a *maximal rationally chain connected fibration* (*MRCC-fibration*, for short) if $\pi' : X \dashrightarrow W'$ is any RCC-fibration then there is a rational map $\tau : W' \dashrightarrow W$ such that $\pi = \pi' \circ \tau$. Moreover, we say that π is a *maximal rationally connected fibration* (*MRC-fibration*, for short) if $\pi_0^{-1}(z)$ is a rationally connected variety for general $z \in Z_0$.

Theorem 2.3 ([F, Theorem 10.4])**.** *Let X be a normal quasi-projective variety and B a boundary $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. In this case, we can construct a projective birational morphism $f : Y \to X$ from a normal quasi-projective variety Y with the following properties.*

(i) *Y is $\mathbb{Q}$-factorial.*
(ii) *$a(E, X, B) \leq -1$ for every f-exceptional divisor E on Y.*

(iii) *We put*

$$B_Y = f_*^{-1}B + \sum_{E:f\text{-}exceptional} E.$$

Then (Y, B_Y) *is dlt and*

$$K_Y + B_Y = f^*(K_X + B) + \sum_{a(E,X,B)<-1} (a(E,X,B)+1)E.$$

In particular, if (X, B) *is lc, then* $K_Y + B_Y = f^*(K_X + B)$. *Moreover, if* (X, B) *is dlt, then we can make* f *small, that is,* f *is an isomorphism in codimension one.*

3. Uniruledness

Proof of Theorem 1.3. By the assumptption, it holds that $(K_X+A)|_A = K_A$. This implies that X has only log terminal singularities around A by the inversion of adjunction ([KoM, Theorem 5.50]). We take a birational map $\varphi : Y \to X$ as in Theorem 2.3 for $(X, 0)$. Then φ is isomorphic around A by $\mathbb{Q}$-factoriality and log terminalicity around A. We may assume that X is $\mathbb{Q}$-factorial log terminal variety by replacing X with Y. From the uniruledness and canonicalicity of A, we see that $\kappa(K_A) = -\infty$

This implies that

$$H^0(X, m(K_X + A) - A) \simeq H^0(X, m(K_X + A)) \tag{1}$$

for a sufficiently large and divisible positive integer m.

Claim 3.1. *It holds that* $H^0(X, m(K_X + A)) = 0$.

Proof of Claim 3.1. If there exists a positive integer m such that

$$H^0(X, m(K_X + A)) \neq 0,$$

A is contained in the base locus of the complete linear system $|m(K_X + A)|$ by (1). Then there exist an effective $\mathbb{Z}$-divisor D_m and a positive integer l such that

$$m(K_X + A) \sim_{\mathbb{Z}} D_m + lA \text{ and } \mathrm{Supp}A \not\subseteq \mathrm{Supp}D_m$$

Since A is semi-ample, there exists a positive integer k such that $|kA|$ is free. We take an effective $\mathbb{Z}$-divisor $B_k \in |kA|$ such that $\mathrm{Supp}A \not\subseteq \mathrm{Supp}B_k$. Thus it holds that

$$km(K_X + A) \sim_{\mathbb{Z}} kD_m + lB_k.$$

This is contradiction from (1). Hence we see that $H^0(X, m(K_X+A)) = 0$ for a sufficiently large and divisible positive integer m. □

We take an effective $\mathbb{Q}$-Cartier divisor H such that $H \sim_{\mathbb{Q}} A$ and (X, H) is klt. Since H is big, $K_X + H$ is not pseudo-effective by the non-vanishing theorem ([BCHM, Theorem D]) and Claim 3.1. Thus we can work some minimal model program and get a Mori fiber space for (X, H) by [BCHM, Corollary 1.3.3]. Hence X is uniruled.

□

4. Rationally connecterdness

Definition 4.1. Let X be a normal variety and A $\mathbb{R}$-Cartier $\mathbb{R}$-Weil divisor on X. We say that A is *strictly nef around* A if there exists a Zariski open set $U \subseteq X$ such that $\mathrm{Supp}A \subseteq U$ and it holds that $(C.A) > 0$ for any proper curve $C \subseteq X$ such that $C \cap U \neq \emptyset$.

Lemma 4.2. *Let X be a normal projective uniruled variety and A Cartier divisor on X. Suppose that A is strictly nef around A and A is a rationally connected variety with only log terminal singularities. If X has $\mathbb{Q}$-factorial and Cohen–Macaulay around A, then X is rationally connected.*

Proof. By the same arguments of proof of Theorem 1.3, we may assume that X is a $\mathbb{Q}$-factorial variety with only log terminal singularities. We take a maximal rationally chain connected fibration $\pi : X \dashrightarrow W$. Then π is a maximal rationally connected fibration by [HM, Corollary 1.5 (2)]. Hence we see that W is not uniruled by [GHS, Corollary 1.4] and $\dim W < \dim X$ by the uniruledness of X. As A is strictly nef around A, $\mathrm{Supp}A$ dominates W. This implies that W is a point from rationally connectedness of A. Thus we see that X is a rationally connected varieties by [HM, Corollary 1.5 (2)]. □

By Theorem 1.3 and Lemma 4.2, we see Theorem 1.4.

Remark 4.3. *For a singular proper variety X, we take maximal rationally connected fibration W of a smooth model Y of X. Then $X \dashrightarrow W$ may not almost holomorphic (the condition* (1) *in* Definition 2.2*). For example, let X be the projective cone over a smooth cubic curve E in $\mathbb{P}^2$. Of course, X is rationally chain connected but is not rationally connected. We take $\pi : \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-1)) \to E$ as a smooth model of X. Then π is maximal rationally connected fibration. Hence $X \dashrightarrow E$ is a linear projection from the vertex. This is not almost holomorphic. So we have to treat MRC-fibration delicately for a singular variety.*

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Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan.

E-mail address: gongyo@ms.u-tokyo.ac.jp