ON NON-RATIONAL LOG SURFACES WITH NEF AND NON-ABUNDANT ANTI-LOG CANONICAL DIVISORS

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Abstract. We classify the non-rational klt log surface pairs with nef and non-abundant anti-log canonical divisors.

1. Introduction

In this paper, we work over the complex number field $\mathbb{C}$. We start by some basic definitions. We will make of the standard notation and definitions as in [B], [Har], and [KoM].

Definition 1.1. We call that (X, Δ) is a *log pair* if X is a normal variety, Δ is an effective $\mathbb{Q}$-divisor and $K_X + \Delta$ is $\mathbb{Q}$-Cartier divisor. We call that (X, Δ) is a *surface pair* if dim $X = 2$ and (X, Δ) is a log pair. A *non-rational* log pair (X, Δ) means that X is not rational variety. Moreover, we define some singularities of pairs. We call (X, Δ)

(1) a *terminal pair* if $\mathrm{discrep(X, \Delta)} > 0$,
(2) a *kawamata log terminal* (*klt*, for short) pair if $\mathrm{discrep(X, \Delta)} > -1$ and $\llcorner \Delta \lrcorner = 0$, and
(3) a *log canonical* (*lc*, for short) pair if $\mathrm{discrep(X, \Delta)} \geq -1$.

Definition 1.2. Let (X, Δ) be a log pair.

(1) $\kappa^{-1}(X, \Delta) := \kappa(-(K_X + \Delta))$ is called the *anti-log Kodaira dimension* of (X, Δ).
(2) $\nu^{-1}(X, \Delta) := \nu(-(K_X + \Delta))$ is called the *numerical anti-log Kodaira dimension* of (X, Δ).

We call that $-(K_X + \Delta)$ is *abundant* if it satisfies that $\kappa^{-1}(S, \Delta) = \nu^{-1}(S, \Delta)$, and $-(K_X + \Delta)$ is *non-abundant* if it satisfies that $\kappa^{-1}(S, \Delta) \neq \nu^{-1}(S, \Delta)$

T. Bauer and T. Peternell prove the following theorem:

Theorem 1.3 (cf. [BP]). *Let X be a smooth projective surface with $-K_X$ nef and non-abundant. Then X is one of the following:*

Case A) *X is $\mathbb{P}^2$ blown up in 9 points in sufficiently general position (possibly infinitely near) or*

Case B) *$X = \mathbb{P}(E)$, E a rank 2 vector bundle over an elliptic curve which is defined by an extension $0 \to \mathcal{O} \to E \to L \to 0$ with L a line bundle of degree 0 and either*

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B.1) $L = \mathcal{O}$ *and the extension is non-split or*
B.2) L *is not torsion.*

We want to investigate the surfaces as in Proposition 1.3 for log surfaces (necessary not smooth). We prove the following theorem:

Main Theorem 1.4. *Let (S, Δ) be a non-rational projective klt surface pair such that $-(K_S + \Delta)$ is nef and non-abundant. Then there exists an elliptic curve Z and rank 2 vector bundle E on Z such that $S \simeq \mathbb{P}_Z(E)$ and that is defined by an extension $0 \to \mathcal{O}_Z \to E \to L \to 0$ with L a line bundle of degree 0 and either*

(i) $L = \mathcal{O}_Z$ and the extension is non-split or
(ii) L is not torsion.

Moreover, Δ is as follows:

(i) $\Delta = \alpha C$ for some $\alpha \in [0, 1)$, where C is the unique member in $| - K_S|$, or
(ii) $\Delta = \alpha_1 C_1 + \alpha_2 C_2$ for some $\alpha_1, \alpha_2 \in [0, 1)$, where $C_1 + C_2$ is the unique member in $| - K_S|$

In particular, S is smooth and Δ is simple normal crossings.

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2. Preliminaries and Lemmas

In this section, we introduce the notation and some lemmas for the proof of Main Theorem 1.4.

Definition 2.1. Let X be a normal projective variety.

(1) X is called *FT* (*Fano type*) if there is a $\mathbb{Q}$-divisor Δ such that (X, Δ) is a klt log Fano pair i.e. (X, Δ) is a klt pair and $-(K_X + \Delta)$ is ample.
(2) X is called 0-type if there is a $\mathbb{Q}$-divisor Δ such that (X, Δ) is a klt pair and $K_X + \Delta \equiv 0$.

Definition 2.2. Let (X, Δ) be a log pair. We call that (X, Δ) *have $\mathbb{Q}$-complement*, if there exists a $\mathbb{Q}$-divisor Δ' on X such that $\Delta \leq \Delta'$, (X, Δ') be an lc pair and $K_X + \Delta' \equiv 0$.

Remark 2.3. *We remark the followings:*

(1) *In dimension 2, we usually use del Pezzo instead of Fano.*
(2) *A FT imply a 0-type.*
(3) *If $-(K_S + \Delta)$ is nef for an lc pair (S, Δ), then there exists an effective $\mathbb{Q}$-divisor D such that $-(K_S + \Delta) \sim_{\mathbb{Q}} D$. (cf.* [P]*)*

Theorem 2.4. [Z, Main Theorem] *Let X be a projective variety and Δ an effective Q-divisor on X such that the pair (X, Δ) is log canonical and $-(K_X + \Delta)$ is nef. Let $f : X \dashrightarrow Y$ be a dominant rational map, where Y is a smooth variety. Then either*

(1) Y *is uniruled; or*
(2) *The Kodaira dimension* $\kappa(Y) = 0$ *Moreover in this case,* f *is semistable in codimension* 1 (*see* [Z, Definition 1]).

Theorem 2.5. [Mi, Theorem 3.1] *Let* Z *be a smooth complete curve over* $\mathbb{C}$*,* E *rank* r *vector bundle on* Z*, and* $\pi : \mathbb{P}_Z(E) \to Z$ *the associated projective bundle. Then the following two condition are equivalent:*
(1) E *is semistable.*
(2) *Every effective divisor on* $\mathbb{P}_Z(E)$ *is nef.*

Lemma 2.6. [P, 8.2.2] *Let* $f : X \to Z$ *be a contraction from a projective surface onto a curve of genus* $g \geq 1$*. Assume that* $K_X + \Delta$ *is lc and* $-(K_X + \Delta)$ *is nef. Assume that the general fiber of* f *is a smooth rational curve. Then no components of* Supp Δ *are contained in fibers.*

3. Classification for non-rational log surfaces with nef and non-abundant anti-log canonical divisors

In this section, we determine the numerical invariants for anti-log canonical divisors of the surfaces as in Main Theorem 1.4 and prove Main Theorem 1.4.

Theorem 3.1. (*cf.* [F2] *and* [Ka]) *Let* (X, B) *be a klt pair and* $\pi : X \to S$ *a proper morphism onto a variety* S*. Assume the following conditions:*
(a) H *is a* π*-nef* $\mathbb{Q}$*-Cartier divisor on* X*,*
(b) $H - (K_X + B)$ *is* π*-nef and* π*-abundant,*
(c) $\kappa(X_\eta, (aH - (K_X + B))_\eta) \geq 0$ *and* $\nu(X_\eta, (aH - (K_X + B))_\eta) = \nu(X_\eta, (H - (K_X + B))_\eta)$ *for some* $a \in \mathbb{Q}$ *with* $a > 1$*, where* η *is the generic point of* S*,*

Then H *is* π*-semiample.*

Proposition 3.2. *For a projective klt surface pair* (S, Δ) *such that* $-(K_S + \Delta)$ *is nef, the following conditions are equivalent:*
(1) $-(K_S + \Delta)$ *is non-abundant,*
(2) $\kappa^{-1}(S, \Delta) = 0$ *and* $\nu^{-1}(S, \Delta) = 1$*, and*
(3) $-(K_S + \Delta)$ *is not semiample.*

Proof. By Remark 2.3 (3), we see that $\kappa^{-1}(S, \Delta) \geq 0$. And it holds that $\kappa^{-1}(S, \Delta) = 0$ is equivalence to $\nu^{-1}(S, \Delta) = 1$. We see that (1) implies that (2).
In general semiampleness of D implies that $\kappa(D) = \nu(D)$ for every $\mathbb{Q}$-Cartier divisor. It holds that (2) implies (3).
By Theorem 3.1, (3) does not holds if (1) does not holds. We finish the proof of Proposition 3.2. □

We can determine the numerical invariants for anti-log canonical divisors of the surfaces as in Main Theorem 1.4. But the above proposition dose not need to prove Main Theorem 1.4.

Proposition 3.3. *Let* (S, Δ) *be an lc surface pair such that* $-(K_S + \Delta)$ *is nef and non-abundant. Then* S *is not a* 0*-type or a Mori dream space in the sense of Hu and Keel* (*cf.* [HK]).

Proof. If S is a Mori dream space, every nef divisor on S is semiample. If S is a 0-type, every effective nef divisor on S is semiample by the log abundance theorem for 2-dimensional klt pairs. $\square$

Proposition 3.4. *Let (S, Δ) be a projective klt surface pair such that $-(K_S + \Delta)$ is nef and $\varphi : (\tilde{S}, \tilde{\Delta}) \to (S, \Delta)$ be the terminalization of (S, Δ), in particular $\tilde{S}$ is smooth. Then $\tilde{S}$ is a rational surface or $\mathbb{P}^1$-fibration over an elliptic curve. i.e. it exists a projective dominant morphism π:$\tilde{S} \to \tilde{Z}$ such that π have any fiber are connected, general fiber are smooth rational curve and $\tilde{Z}$ is an smooth elliptic curve.*

Proof. We may assume that $\tilde{S}$ is not rational. Then $\tilde{S}$ is a $\mathbb{P}^1$-fibration over an some smooth curve $\tilde{Z}$ since $\kappa(K_{\tilde{S}}) = -\infty$. It suffices to show that $\tilde{Z}$ is an elliptic curve. By Theorem 2.4, we see that $\tilde{Z}$ is an elliptic curve. $\square$

Proposition 3.5. *Let (S, Δ) be a projective lc surface pair such that $-(K_S + \Delta)$is nef, $f : (S, \Delta) \to (S', \Delta')$ be a birational $(K_S + \Delta)$-extremal contraction. Then (S', Δ') is an lc weak log del Pezzo pair.*

Proof. Since f is a divisorial contraction, the exceptional set is irreducible curve E. We write

$$K_S + \Delta = f^*(K_{S'} + \Delta') + aE, \ a > 0.$$

Clearly $-(K_{S'}+\Delta')$ is nef. By Remark 2.3 (3) and $E^2 < 0$, we see that $(K_{S'}+\Delta')^2 > 0$. This implies that $-(K_{S'} + \Delta')$ is an lc weak log del Pezzo pair. $\square$

Theorem 3.6. *Let (S, Δ), $(\tilde{S}, \tilde{\Delta})$, π, $\tilde{Z}$ be as in* Proposition 3.4. Assume that $\tilde{Z}$ is elliptic curve. Then it satisfy the followings:

(1) (S, Δ) is a terminal pair, in particular $(S, \Delta) = (\tilde{S}, \tilde{\Delta})$ (up to isomorphic).
(2) S have a $\mathbb{P}^1$-bundle structure over $Z := \tilde{Z}$ i.e. it exists a vector bundle E of rank 2 on Z such that $S \simeq \mathbb{P}_Z(E)$ over Z.

Proof. We run the log minimal model program starting from $(\tilde{S}, \tilde{\Delta})$. Since $\tilde{S}$ is not rational and a klt weak log del pezzo pair is rational, the first contraction in this log minimal model program is fiber type by Proposition 3.5. In particular, this contraction coincide with π. Since S have only rational singularities, it induces the algebraic fiber space $\pi' : S \to \tilde{Z}$ such that $\pi = \varphi \circ \pi'$. But, since $\rho(S/Z) = 1$, φ is an isomorphism. This implies that it satisfies the condition (1). For proving the condition (2), it suffices to show π is K_S-extremal contraction. Now, if F is a general fiber of π, then it holds that $-K_S.F = -(K_S + F).F = \deg(-K_F) = 2$. This implies that π is K_S-extremal contraction. $\square$

Theorem 3.7. *Let (S, Δ), π, Z, E be as in* Theorem 3.6. Then E is defined by an extension $0 \to \mathcal{O}_Z \to E \to L \to 0$ with L a line bundle of degree 0 and either

(i) $L = \mathcal{O}_Z$ and the extension is non-split or
(ii) L is not torsion.

Moreover, Δ is as follows:

(i) $\Delta = \alpha C$ for some $\alpha \in [0, 1)$, where C is unique member in $|-K_S|$, or

(ii) $\Delta = \alpha_1 C_1 + \alpha_2 C_2$ for some $\alpha_1, \alpha_2 \in [0, 1)$, where $C_1 + C_2$ is unique member in $|-K_S|$

Proof. We see the following claims:

Claim 3.8. *E is a semistable vector bundle over Z.*

Proof of Claim 3.8. By Kawamata-Shokurov base point free theorem, $-(K_S+\Delta)$ is not big. Let F be a general fiber of π. Then $-(K_S+\Delta).F \neq 0$. This implies that $[-(K_S+\Delta)]$ generates the edge which is different from the edge generated by F in the nef cone of S. Assume by contradiction that E is not semistable. By Theorem 2.5, the cone of effective divisors on S does not coincide with the nef cone. Since $[-(K_S+\Delta)]$ generates the edge which is different from the edge generated by F in nef cone, $[-(K_S+\Delta)]$ is contained in the relative interior of the cone of effective divisors on S. This implies that $[-(K_S+\Delta)]$ is big since the big cone is the relative interior of the cone of effective divisors. This is contradiction. We finish the proof of Claim 3.8. □

Claim 3.9. *$-K_S$ is nef.*

Proof of Claim 3.9. Since E is semistable, the cone of effective divisors on S coincides with the nef cone of S by Theorem 2.5. From this, $-K_S$ is nef. □

Claim 3.10. *$-K_S$ is not semiample.*

Proof of Claim 3.10. Assume by contradiction that $-K_S$ is semiample. If $-K_S$ is big, S is weak del Pezzo, in particular, is rational. Hence $-K_S$ is not big. If $\kappa(-K_S) = 0$, it holds that $K_S \sim_{\mathbb{Q}} 0$ by the semiampleness of $-K_S$. But S is an uniruled surface. We see that $\kappa(-K_S) \neq 0$. Hence we may assume that $\kappa(-K_S) = 1$. Let $\phi := \phi_{|-mK_S|} : S \to T$ for $m \gg 0$. Then T is a smooth rational curve by uniruledness of S. By Remark 2.3 (3), there exists an effective divisor D such that $-(K_S+\Delta) \sim_{\mathbb{Q}} D$. By applying Lemma 2.6 to π and the klt pair $(S, \Delta + \varepsilon D)$ for some $\varepsilon \ll 1$, each component of $\Delta + D$ is not rational curve. Since $\Delta + D$ is $\mathbb{Q}$-linearly trivial over T. Each component of $\Delta + D$ does not dominant T. Thus each component of Δ is a smooth fiber of ϕ, in particular, D is semiample. This contradicts the assumption of Theorem 3.7. We finish the proof of Claim 3.10. □

By Theorem 1.3, the structure of E is as in Theorem 3.7. We see that there exists the unique member in $|-mK_S|$ for m as in (i) or (ii) (cf. [BP, Remark 1.7]). Thus the structure of Δ is as in Theorem 3.7. □

We finish the proof of Main Theorem 1.4.

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