EXAMPLE OF A PLT PAIR OF LOG GENERAL TYPE WITH INFINITELY MANY LOG MINIMAL MODELS

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Conjecture 0.1. *Let $\pi\colon X \to U$ be a projective morphism of normal quasi-projective varieties, where X has dimension d. Suppose (X, Δ) be $\mathbb{Q}$-factorial purely log terminal pair over U, $K_X + \Delta$ is big over U. Then the set of isomorphism classes*

$$\{\, \phi : X \dashrightarrow Y \mid \phi \text{ is the log minimal model over } U \text{ of } (X, \Delta)\}$$

is finite.

Remark 0.2. *This conjecture for klt pair is true or in the case of $K_X + \Delta$ is log big is true by* [BCHM].

But this conjecture is not true for plt pair in general.

Example 0.3. *Let S be a $K3$ surface with infinitely many (-2)-curve (cf.* [Kov]*) and $S \subset \mathbb{P}^N$ some projectively normal embedding. Let X_0 be the cone over it and $\phi\colon X \to X_0$ the blow-up at the vertex. Then the linear projection $X_0 \dashrightarrow S$ from the vertex is decomposed as follows:*

(1)
$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\phi} & & \searrow^{\pi} & \\ X_0 & & & & S. \end{array}$$

Let $H' \subset X_0$ be a sufficiently ample divisor which does not contain the origin and $K_{X_0} + H'$ is ample. Let $E \subset X$ be the ϕ-exceptional divisor, and let H be the proper transform of H' in X. Then the pair $(X, \Delta = E + H)$ is purely log terminal. Since $K_X + E + H = \phi^*(K_{X_0} + H')$ (cf. Propositon *4.38 in* [F]*) is nef and big, (X, Δ) is plt and 3-fold of log general type such that $K_X + \Delta$ is nef.*
Let $\{C_i\}$ be infinitely many (-2)-curves on E.
We claim that

Claim 0.4. $\mathbb{R}_{\geq 0}[C_i] \subseteq \overline{NE}(X)$ *is an extremal ray with $(K_X + \Delta).C_i = 0$ and $(K_X + \Delta + \delta_i D_i).C_i < 0$, where D_i is $\pi^*(\pi(C_i))$ and δ_i is a surfficiently small positive number.*
Morever, let ϕ_{C_i} be extremal contraction associated to $\mathbb{R}_{\geq 0}[C_i]$. Then ϕ_{C_i} is the $(K_X + \Delta + \delta_i D_i)$-flipping contraction and the $(K_X + \Delta)$-flopping contraction.

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Proof. It holds that $(K_X+\Delta).C_i = 0$ by $(K_X+\Delta)_{|E} = K_E$ and $(K_X+\Delta+\delta_i D_i).C_i < 0$ by $C_i^2 = -2$.
We prove that $\mathbb{R}_{\geq 0}[C_i] \subseteq \overline{NE}(X)$ is an extremal. If there is pseudoeffective curves $G_1,\ G_2 \in \overline{NE}(X)$ such that $[C_i] = [G_1] + [G_2]$, we can see $H.G_j = 0$. So it holds that $\mathrm{Supp}(G_j) \subseteq E$. We take semiample divisor L_i on S such that L_i is a supporting divisor of the extremal ray $\mathbb{R}_{\geq 0}[C_i]$, i.e. L_i satisfies $L_i.C_i = 0$ and $L_i.G > 0$ for any pseudoeffective curve $[G] \in \overline{NE}(E)$ such that $[G] \in \mathbb{R}_{\geq 0}[C_i]$ on E. We identify E with S. Let $\mathscr{L}_i$ be a pullbuck of L_i by π. We see that $\mathscr{L}_i.G_j = \mathscr{L}_{i|E}.G_j = L_i.G_j = 0$. So there exists a nonnegative number α_j such that $G_j = \alpha_j C_i$. We also see that $[C_i] = \{C_i\}$ and ϕ_{C_i} is small contraction. □

Now, since ϕ_{C_i} is the $(K_X + \Delta + \delta_i D_i)$-flipping contraction, its log flip $X \dashrightarrow X_i$ exists, which is the log flop for $K_X + \Delta$. We see that log flop $f_i : (X, \Delta) \dashrightarrow (X_i, \Delta_i)$ is log minimal model, where Δ_i is the strict transform of Δ on X_i. But it holds that $f_i \not\simeq f_j$ $(i \neq j)$.

This example is inspired by that of Hacon and McKernan in Lazić's paper (cf. [L, Theorem A.6]).

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